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The onset of Rayleigh–Bénard convection in non-planar oscillatory flows

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The onset of thermal convection in the presence of an oscillatory, non-planar shear flow is investigated on a linear basis. For the case of planar oscillations, the basic shear has no effect upon the value of the critical Rayleigh number but does act as a pattern selection mechanism. For the non-planar case, when there are two horizontal components of the basic velocity, the same result is true if the components are either in phase or directly out of phase. For the general case, however, stabilization occurs because convection rolls experience the stabilizing effects of shear regardless of their orientation. The results are obtained both by expansion in terms of the amplitude of the oscillating flow and in terms of its frequency, assuming the frequency to be small. The degree of stabilization increases with the Prandtl number. Pattern selection still occurs with non-planar oscillations.

1. Introduction

There have been many investigations concerning how a periodic modulation with time of the temperature difference across a horizontal fluid layer initially at rest affects the onset of Rayleigh–Bénard convection. Many of the results obtained on the basis of both linear theory and energy theory have been reviewed by Davis (1976). For the case, say, of modulation of the temperature on only the lower boundary, use of linear Floquet theory predicts that the critical Rayleigh number (Ra_c) will be greater than the case without modulation, with the maximum effect occurring in general near a Prandtl number of unity and in the quasi-steady limit (non-dimensional frequency $\beta^2 \rightarrow 0$). The stabilization predicted by theory was observed experimentally, at least at relatively high frequencies, first by Finucane & Kelly (1976), and more recently by others as summarized by Donnelly (1990). At low frequencies, however, the linearized result is invalid for two reasons. First, transient convection was observed during part of each cycle by Finucane & Kelly (1976) so that the use of Floquet theory, which concerns the net growth or decay of a single disturbance over a cycle, no longer describes realistically the response of the system. This aspect of the problem has been discussed in considerable detail by Barenghi & Jones (1989) for the analogous problem of modulated Taylor–Couette flow. Second, hexagonal convection can occur at subcritical values of the Rayleigh number which can invalidate the linear results; see Roppo, Davis & Rosenblat (1984) and Meyer, Cannell & Ahlers (1992).

Corresponding results for the case when the fluid oscillates, say, about a zero mean velocity and with constant wall temperature, are apparently not available in the literature, although such oscillations can occur readily in applications. For the case of a steady, fully developed, horizontally unbounded, unidirectional shear flow, it is well-known that the shear has no effect upon Ra_c. This result occurs because the shear stabilizes all buoyancy-driven disturbances except for those with zero streamwise wavenumber, i.e. disturbances which are spanwise periodic; see e.g. Fujimura & Kelly (1988) and the references listed there. For such disturbances, usually called longitudinal rolls, the basic flow does not enter into the equations governing the cross-stream disturbance velocity components and the temperature disturbance that give rise to the eigenvalue problem for Ra_c, which therefore has the same value for longitudinal rolls as the case without shear. The shear acts essentially as a pattern selection mechanism because a preferred wavenumber vector occurs for the case with shear. For certain applications (e.g. situations involving a phase change), this can be important because a very well-defined state of convection occurs for Ra somewhat in excess of Ra_c rather than the rather jumbled state typical of the case without shear or forcing. In some situations, it might be easier to impose a periodic shear on the system than a steady one. For the case of a unidirectional but time-wise-periodic shear, the same result should be expected at sufficiently low values of β^2. Kelly & Thompson (1988) have shown that the same result also occurs at higher values of β^2; their results are contained in the present ones and, in an appendix to this paper, the disturbance energy equation is examined in order to clarify the physical basis for this result.

For the case when non-planar oscillations occur, i.e. there are two components of the basic flow in the horizontal plane which are out of phase, the present analysis indicates that stabilization is predicted on the basis of linear theory. The effect is a maximum for high Prandtl number fluids as the frequency tends to zero, at least for the case of a shear due to a single oscillating wall. For the present case, it will be argued that the quasi-steady result is indeed meaningful within the context of linear theory. Whether or not the linear result is meaningful requires further analysis of the nonlinear problem.

2. Formulation

We assume that the Oberbeck–Boussinesq equations are applicable, namely,

$$\boldsymbol{V}^*_{t^*} + \boldsymbol{V}^* \cdot \nabla \boldsymbol{V}^* = -\frac{1}{\rho_0}\nabla p^* - \alpha \boldsymbol{g}(T^* - T^*_0) + \nu_0 \nabla^2 \boldsymbol{V}^*, \tag{2.1a}$$

$$T^*_{t^*} + \boldsymbol{V}^* \cdot \nabla T^* = \kappa_0 \nabla^2 T^*, \tag{2.1b}$$

$$\nabla \cdot \boldsymbol{V}^* = 0, \tag{2.1c}$$

where $\boldsymbol{V}^*$ is the velocity, p^* the pressure, T^* the temperature, g gravity, ν_0 kinematic viscosity, κ_0 diffusivity, α the expansion coefficient, and ρ_0 and T^*_0 are reference density and temperature, respectively. Let x^* and y^* denote dimensional spatial variables in the plane of the fluid layer and z^* be distance normal to this layer.

The basic flow is periodic in time t^* with frequency ω^* and has two components so

$$\boldsymbol{V}^*(z^*, t^*) = \boldsymbol{i}_{x^*} U^*(z^*, t^*) + \boldsymbol{i}_{y^*} V^*(z^*, t^*). \tag{2.2}$$

The fluctuating flow is considered to be either of the Couette type (case I) driven by wall oscillations of the form

$$U^*(0, t^*) = \bar{U}^*_0 \cos \omega^* t^*, \quad U^*(h, t^*) = \bar{U}^*_1 \cos(\omega^* t^* + \sigma), \tag{2.3a, b}$$

$$V^*(0, t^*) = \bar{V}^*_0 \cos(\omega^* t^* + \gamma_0), \quad V^*(h, t^*) = \bar{V}^*_1 \cos(\omega^* t^* + \gamma_1), \tag{2.4a, b}$$

where h is the depth of the fluid layer and σ, γ_0, and γ_1 are phase angles, or of the Poiseuille type (case II) driven by oscillating pressure gradient of the form

$$\frac{\partial p^*}{\partial x^*} = \frac{(\Delta \bar{p}^*)_{x^*}}{L_{x^*}} \cos \omega^* t^*, \quad \frac{\partial p}{\partial y^*} = \frac{(\Delta \bar{p}^*)_{y^*}}{L_{y^*}} \cos \{\omega^* t^* + \gamma_2\}, \tag{2.5a, b}$$

where L_{x^*} and L_{y^*} are characteristic lengths in the x^*- and y^*-directions and $\Delta \bar{p}^*$ denotes a pressure difference. If we now introduce non-dimensional variables of the form

$$(x, y, z) = (x^*/h, y^*/h, z^*/h), \quad t = \omega^* t^*, \tag{2.6}$$

then $U^*(z, t)$ say, for case I, is given by the solution of

$$2\beta^2 \, \partial U^*/\partial t = \partial^2 U^*/\partial z^2, \tag{2.7a}$$

$$U^*(0, t) = \bar{U}_0^* \cos t, \quad U^*(1, t) = \bar{U}_0^* A_x \cos (t + \sigma), \tag{2.7b}$$

where $A_x = \bar{U}_1^*/\bar{U}_0^*$ and $\beta^2 = \omega^* h^2/2\nu_0$. We will be interested mainly in the cases when $A_x = 0$ or 1 and $\sigma = 0$ or π. Let

$$U(z, t) = U^*/\bar{U}_0^* = \tfrac{1}{2}\phi_0(z)\, e^{it} + \tfrac{1}{2}\tilde{\phi}_0(z)\, e^{-it} + \tfrac{1}{2}A_x \phi_1(z)\, e^{i(t+\sigma)} + \tfrac{1}{2}A_x \tilde{\phi}_1(z)\, e^{-i(t+\sigma)}, \tag{2.8}$$

where $(\tilde{\ })$ denotes the complex conjugate of any quantity. The functions ϕ_0 and ϕ_1 are given by

$$\phi_0(z) = \frac{e^{(1+i)\beta(1-z)} - e^{-(1+i)\beta(1-z)}}{e^{(1+i)\beta} - e^{-(1+i)\beta}} \tag{2.9}$$

and

$$\phi_1(z) = \phi_0(1-z). \tag{2.10}$$

A similar solution holds for $V^*(z, t)$ with $\bar{V}_0^*$ serving as the characteristic velocity, namely,

$$V(z, t) = V^*/\bar{V}_0^* = \tfrac{1}{2}\phi_0(z)\, e^{i(t+\gamma_0)} + \tfrac{1}{2}\tilde{\phi}_0\, e^{-i(t+\gamma_0)} + \tfrac{1}{2}A_y \phi_1(z)\, e^{i(t+\gamma_1)} + \tfrac{1}{2}A_y \tilde{\phi}_1(z)\, e^{-i(t+\gamma_1)}, \tag{2.11}$$

where $A_y = \bar{V}_1^*/\bar{V}_0^*$.

For case II when the flow arises due to the oscillating pressure gradients (2.5*a, b*), $U^*(z, t)$ is given by the solution of

$$2\beta^2 \, \partial U^*/\partial t = -\bar{U}_2^* \cos t + \partial^2 U^*/\partial z^2 \tag{2.12a}$$

$$U^*(0, t) = U^*(1, t) = 0, \tag{2.12b}$$

where $\bar{U}_2^* = (\Delta \bar{p})_x h^2/\mu L$. We let

$$U(z, t) = U^*/\bar{U}_2^* = \tfrac{1}{2}\phi_2(z)\, e^{it} + \tfrac{1}{2}\tilde{\phi}_2(z)\, e^{-it}, \tag{2.13}$$

where

$$\phi_2(z) = \frac{-i}{2\beta^2}\left\{\frac{e^{(1+i)\beta z} - e^{-(1+i)\beta z}}{e^{(1+i)\beta} - e^{-(1+i)\beta}}\right\} - \frac{i}{2\beta^2}\left\{\frac{e^{(1+i)\beta(1-z)} - e^{-(1+i)\beta(1-z)}}{e^{(1+i)\beta} - e^{-(1+i)\beta}}\right\} + \frac{i}{2\beta^2}. \tag{2.14}$$

Naturally a similar solution holds for $V^*(z, t)$ with $\bar{V}_2^* = (\Delta \bar{p})_y h^2/\mu L$.

Assuming that viscous dissipation is negligible, the basic temperature $T^*(z^*)$ is independent of the flow oscillations and is given by

$$T^*(z^*) = T^*(0) - (z^*/h)\, \Delta \bar{T}, \tag{2.15}$$

where $\Delta \bar{T} = T^*(0) - T^*(h) = $ constant.

Now let

$$V^* = \boldsymbol{i}_x \{\bar{U}_c^* U + (\kappa_0/h)\, u\} + \boldsymbol{i}_y \{\bar{V}_c^* V + (\kappa_0/h)\, v\} + \boldsymbol{i}_z (\kappa_0/h)\, w, \tag{2.16a}$$

$$T = T^*(z^*) + \Delta \bar{T} \theta, \tag{2.16a}$$

where u, v, w and θ depend on x, y, z, t. After substituting (2.16*a*, *b*) into (2.1*a*–*c*) and linearizing about the basic state, the following two equations can be obtained by manipulation for w and θ:

$$\left\{2\beta^2\frac{\partial}{\partial t}+Re_x\,U\frac{\partial}{\partial x}+Re_y\,V\frac{\partial}{\partial y}-\nabla^2\right\}\nabla^2 w-Re_x\frac{\partial^2 U}{\partial z^2}\frac{\partial w}{\partial x}-Re_y\frac{\partial^2 V}{\partial z^2}\frac{\partial w}{\partial y}=Ra\,\nabla_\perp^2\,\theta, \quad (2.17a)$$

$$\left\{2\beta^2 Pr\frac{\partial}{\partial t}+Re_x Pr\,U\frac{\partial}{\partial x}+Re_y Pr\,V\frac{\partial}{\partial y}-\nabla^2\right\}\theta=w, \quad (2.17b)$$

where

$$Re_x=\bar{U}_c^*\,h/\nu_0,\quad Re_y=\bar{V}_c^*\,h/\nu_0,\quad Pr=\nu_0/\kappa_0,\quad Ra=\alpha g\Delta\bar{T}h^3/\nu_0\,\kappa_0, \quad (2.18)$$

if $\bar{U}_c^*=\bar{U}_0^*$ (case I) or $\bar{U}_c^*=\bar{U}_2^*$ (case II), etc. We consider the case of rigid, isothermal surfaces, and so the boundary conditions are

$$w=\partial w/\partial z=\theta=0\quad\text{at}\quad z=0,1. \quad (2.19)$$

Now let

$$w(x,y,z,t)=W(z,t)\,\mathrm{e}^{\mathrm{i}(k_x x+k_y y)}+\text{c.c.}, \quad (2.20a)$$

$$\theta(x,y,z,t)=\Theta(z,t)\,\mathrm{e}^{\mathrm{i}(k_x x+k_y y)}+\text{c.c.}, \quad (2.20b)$$

where c.c. denotes complex conjugate.

The linearized stability problem consists of solving

$$\left\{2\beta^2\frac{\partial}{\partial t}+\mathrm{i}k_x Re_x\,U+\mathrm{i}k_y Re_y\,V-\left(\frac{\partial^2}{\partial z^2}-k^2\right)\right\}\left(\frac{\partial^2}{\partial z^2}-k^2\right)W$$

$$-\mathrm{i}k_x Re_x\frac{\partial^2 U}{\partial z^2}W-\mathrm{i}k_y Re_y\frac{\partial^2 V}{\partial z^2}W=-k^2 Ra\Theta, \quad (2.21a)$$

$$\left\{2\beta^2 Pr\frac{\partial}{\partial t}+\mathrm{i}k_x Re_x Pr\,U+\mathrm{i}k_y Re_y Pr\,V-\left(\frac{\partial^2}{\partial z^2}-k^2\right)\right\}\Theta=W, \quad (2.21b)$$

subject to the boundary conditions

$$W=\partial W/\partial z=\Theta=0\quad\text{at}\quad z=0,1. \quad (2.22)$$

3. Expansion in terms of Reynolds number

In order to determine Ra_c, it should be realized that it is a function of many parameters, namely, $Ra_c=Ra_c\,(Re_x, Re_y, Pr, \beta, A_x, A_y, \sigma, \gamma_0, \gamma_1, k_x, k_y)$ and so some limiting cases should be examined prior to a general analysis in order to provide a framework. We first consider the case of small-amplitude oscillations. We define $Re_x = Re$ and $\lambda = Re_y/Re$ and expand in terms of Re with λ fixed, i.e. we expand as

$$W(z,t)=W_0(z)+Re\,W_1(z,t)+Re^2W_2(z,t)+\ldots, \quad (3.1a)$$

$$\Theta(z,t)=\Theta_0(z)+Re\,\Theta_1(z,t)+Re^2\Theta_2(z,t)+\ldots, \quad (3.1b)$$

$$Ra=Ra_c=Ra_{c,0}+Re\,Ra_1+Re^2Ra_2+\ldots. \quad (3.1c)$$

Thus, as $Re\rightarrow 0$, Ra is equal to the critical Rayleigh number for the case without shear ($Ra_{c,0}$) and we wish to see how this critical value is affected by the oscillations. It can be argued that the change in Ra_c should not depend on the sign of Re, which is tantamount to a change in phase of π, and so $Ra_1=0$ (as can be demonstrated at $O(Re)$ explicitly).

As $Re\rightarrow 0$, we have the standard equations governing Rayleigh–Bénard convection when $Ra=Ra_c$, namely,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)^2W_0-k^2Ra_{c,0}\,\Theta_0=0,\quad \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)\Theta_0+W_0=0, \quad (3.2a,b)$$

where $k^2 = k_x^2 + k_y^2 = k_{c,0}^2$. We will set $k^2 = k_{c,0}^2 \approx (3.117)^2$ in the following. At $O(Re^1)$, the governing equations are

$$\left\{2\beta^2 \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\right\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) W_1 + k^2 Ra_{c,0}\, \Theta_1$$

$$= -\mathrm{i}(k_x U + \lambda k_y V)\left(\frac{\partial^2}{\partial z^2} - k^2\right) W_0 + \mathrm{i}\left(k_x \frac{\partial^2 U}{\partial z^2} + \lambda k_y \frac{\partial^2 V}{\partial z^2}\right) W_0, \quad (3.3a)$$

$$\left\{2\beta^2 Pr \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\right\} \Theta_1 - W_1 = -\mathrm{i} Pr\, (k_x U + \lambda k_y V)\, \Theta_0, \quad (3.3b)$$

Notice that we can make the non-homogeneous terms zero for $V = 0$, $k_x = 0$ (so $k_y = k_c$). These solutions correspond to longitudinal rolls, whose onset is unaffected by the unidirectional shear.

The boundary conditions (2.22) are imposed at every order. In order to minimize the length of the equations, we now focus on case I when $\Lambda_x = \Lambda_y = 0$, which is sufficient to demonstrate the method of analysis. Results presented later for Case I for $\Lambda_x \neq 0$, $\Lambda_y \neq 0$ and also for Case II have been obtained by use of the same basic approach. We let

$$W_1(z,t) = W_{11}(z)\,\mathrm{e}^{\mathrm{i}t} + \hat{W}_{11}(z)\,\mathrm{e}^{-\mathrm{i}t} + W_{12}(z)\,\mathrm{e}^{\mathrm{i}(t+\gamma_0)} + \hat{W}_{12}(z)\,\mathrm{e}^{-\mathrm{i}(t+\gamma_0)}, \quad (3.4a)$$

$$\Theta_1(z,t) = \Theta_{11}(z)\,\mathrm{e}^{\mathrm{i}t} + \hat{\Theta}_{11}(z)\,\mathrm{e}^{-\mathrm{i}t} + \Theta_{12}(z)\,\mathrm{e}^{\mathrm{i}(t+\gamma_0)} + \hat{\Theta}_{12}(z)\,\mathrm{e}^{-\mathrm{i}(t+\gamma_0)}. \quad (3.4b)$$

Thus, the subscript $(\,)_{11}$ refers to interaction between the Rayleigh–Bénard convection and the oscillatory flow in the x-direction, whereas $(\,)_{12}$ refers to interaction between the convection and the flow in the y-direction. For brevity, the equations for (W_{11}, Θ_{11}), $(\hat{W}_{11}, \hat{\Theta}_{11})$, (W_{12}, Θ_{12}) and $(\hat{W}_{12}, \hat{\Theta}_{12})$ are omitted.

Proceeding now to $O(Re^2)$, we have

$$\left\{2\beta^2 \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\right\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) W_2 + k^2 Ra_{c,0}\, \Theta_2$$

$$= -\mathrm{i}(k_x U + \lambda k_y V)\left(\frac{\partial^2}{\partial z^2} - k^2\right) W_1 + \mathrm{i}\left(k_x \frac{\partial^2 U}{\partial z^2} + \lambda k_y \frac{\partial^2 V}{\partial z^2}\right) W_1 - k^2 Ra_2\, \Theta_0, \quad (3.5a)$$

$$\left\{2\beta^2 Pr \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\right\} \Theta_2 - W_2 = -\mathrm{i} Pr\, (k_x U + \lambda k_y V)\, \Theta_1, \quad (3.5b)$$

and so we seek solutions of the type

$$W_2(z,t) = W_{20}(z) + W_{21}(z)\,\mathrm{e}^{2\mathrm{i}t} + \hat{W}_{21}(z)\,\mathrm{e}^{-2\mathrm{i}t} + W_{22}(z)\,\mathrm{e}^{2\mathrm{i}(t+\gamma_0)} + \hat{W}_{22}(z)\,\mathrm{e}^{-2\mathrm{i}(t+\gamma_0)}, \quad (3.6a)$$

$$\Theta_2(z,t) = \Theta_{20}(z) + \Theta_{21}(z)\,\mathrm{e}^{2\mathrm{i}t} + \hat{\Theta}_{21}(z)\,\mathrm{e}^{-2\mathrm{i}t} + \Theta_{22}(z)\,\mathrm{e}^{2\mathrm{i}(t+\gamma_0)} + \hat{\Theta}_{22}(z)\,\mathrm{e}^{-2\mathrm{i}(t+\gamma_0)}. \quad (3.6b)$$

As far as the determination of Ra_2 is concerned, we only have to consider the steady components W_{20} and Θ_{20} because the equations governing them determine Ra_2 via a solvability condition. After substituting (3.6a,b) into (3.5a,b), we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^2 W_{20} - k^2 Ra_{c,0}\, \Theta_{20} = -\tfrac{1}{2}(k_x^2 + 2\lambda k_x k_y \cos\gamma_0 + \lambda^2 k_y^2)$$

$$\times [\phi_0(\tilde{F}_{11}''' - k^2 \tilde{F}_{11}') + \tilde{\phi}_0(F_{11}''' - k^2 F_{11}') - \tilde{\phi}_0'' F_{11} - \phi_0'' \tilde{F}_{11}] + k^2 Ra_2\, \Theta_0, \quad (3.7a)$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right) \Theta_{20} + W_{20} = -\tfrac{1}{2} Pr\, (k_x^2 + 2\lambda k_x k_y \cos\gamma_0 + \lambda^2 k_y^2)\, (\phi_0 \tilde{G}_{11} + \tilde{\phi}_0 G_{11}), \quad (3.7b)$$

where

$$W_{11} = \mathrm{i}k_x F_{11}, \quad \hat{W}_{11} = \mathrm{i}k_x \tilde{F}_{11}, \quad \Theta_{11} = \mathrm{i}k_x G_{11}, \quad \hat{\Theta}_{11} = \mathrm{i}k_x \tilde{G}_{11}, \tag{3.8a}$$

$$W_{12} = \mathrm{i}\lambda k_y F_{11}, \quad \hat{W}_{12} = \mathrm{i}\lambda k_y \tilde{F}_{11}, \quad \Theta_{12} = \mathrm{i}\lambda k_y G_{11}, \quad \hat{\Theta}_{12} = \mathrm{i}\lambda k_y \tilde{G}_{11}. \tag{3.8b}$$

We noted after (3.3a, b) conditions under which the non-homogeneous terms are zero in those equations and naturally the same conditions cause the non-homogeneous terms in (3.7a,b) to be zero. For these special cases, we can conclude that $Ra_2 = 0$. For general λ, however, the coefficients involving k_x and k_y are non-zero. Say that we express k_x and k_y in terms of the amplitude (k) and angle (θ) of the wavenumber vector, i.e. we let

$$k_x = k\cos\theta, \quad k_y = k\sin\theta. \tag{3.9}$$

We then have that the coefficient appearing in (3.7a,b) is equal to

$$\begin{aligned}(k_x^2 + 2\lambda k_x k_y \cos\gamma_0 + \lambda^2 k_y^2) &= k^2(\cos^2\theta + \lambda^2\sin^2\theta + \lambda\cos\gamma_0 \sin 2\theta) \\ &= k^2\{(\cos\theta + \lambda\cos\gamma_0\sin\theta)^2 + \lambda^2\sin^2\theta\sin^2\gamma_0\} \\ &\stackrel{\text{def}}{=} k^2 K(\theta, \lambda, \gamma_0).\end{aligned} \tag{3.10}$$

For $\lambda = 0$ corresponding to shear in the x-direction only, we can say that (3.10) is equal to zero if $k_x = 0$, which corresponds to longitudinal rolls ($\theta = \pm\frac{1}{2}\pi$).

In order to determine Ra_2, we obtain the solvability condition for (3.7a,b) by multiplying (3.7a) by W_0, (3.7b) by $(-k^2 Ra_{c,0}\Theta_0)$, integrating from $z = 0$ to 1, and adding the results, as follows:

$$\begin{aligned}Ra_2\int_0^1 W_0\Theta_0\,\mathrm{d}z = \tfrac{1}{2}K\int_0^1 [\{\phi_0(\tilde{F}_{11}'' - k^2\tilde{F}_{11}) + \tilde{\phi}_0(F_{11}'' - k^2 F_{11}) \\ - \tilde{\phi}_0'' F_{11} - \phi_0''\tilde{F}_{11}\}W_0 - k^2 Pr\,Ra_{c,0}\{\phi_0\tilde{G}_{11} + \tilde{\phi}_0 G_{11}\}\Theta_0]\,\mathrm{d}z.\end{aligned} \tag{3.11}$$

Divide through by the coefficient of Ra_2 to get

$$Ra_2 = \tfrac{1}{2}K\mathscr{R} \tag{3.12}$$

where $\mathscr{R}$ is the ratio of the integral of the terms in brackets on the right of (3.11) to the integral of $W_0\Theta_0$ and is a function of β and Pr only. Numerical results to be presented shortly indicate that in general $\mathscr{R} \geqslant 0$. As $\beta \to \infty$, $\mathscr{R} \to 0$ because the unsteady Stokes layer is very small in comparison to the depth of the layer. Otherwise, $\mathscr{R} > 0$, which implies, as we now show, that $Ra_2 > 0$ if $\lambda \neq 0, \infty$ (i.e. non-planar oscillations occur) and $\gamma_0 \neq 0$ or π.

Returning to (3.10), note that

$$\partial K/\partial\gamma_0 = -\lambda\sin\gamma_0\sin 2\theta, \tag{3.13}$$

and so K has an extremum for $\gamma_0 = 0$ or π, corresponding to the case when U and V are either in phase or out of phase by 180°. For $\gamma_0 = 0$, $K = 0$ if $\tan\theta = -\lambda^{-1}$, whereas, for $\gamma_0 = \pi$, $K = 0$ if $\tan\theta = \lambda^{-1}$; in either case, longitudinal rolls that are unaffected by shear are possible in a suitably redefined coordinate system. For the other values of γ_0, however, K is clearly greater than zero, leading to the conclusion that non-planar oscillations are stabilizing in general. In view of the above result, it would seem that the largest amount of stabilization occurs when the oscillations are as 'non-planar as possible', i.e. $\gamma_0 = \pm\frac{1}{2}\pi$, as follows also from (3.10). For this choice of γ_0,

$$K = (\cos^2\theta + \lambda^2\sin^2\theta) > 0, \tag{3.14}$$

$$\partial K/\partial\theta = (\lambda^2 - 1)\sin 2\theta, \tag{3.15}$$

and so K is a minimum for $\theta = \pm\frac{1}{2}\pi$ if $\lambda < 1$ and likewise for $\theta = 0, \pi$ if $\lambda > 1$. Thus, a preferred pattern is predicted on the basis of linear theory for all values of λ. For $\lambda < 1$, we have longitudinal rolls with their axes in the x-direction. For $\lambda > 1$, longitudinal rolls with axes in the y-direction are preferred. For $\lambda = 1$ and $\gamma_0 \neq 0, \pi$ or $\pm\frac{1}{2}\pi$, the rolls are oriented at an angle of 45° to the x-axis (but no preferred orientation exists if $\gamma_0 = \pm\frac{1}{2}\pi$).

So far, we have considered only the case when $\Lambda_x = \Lambda_y = 0$, i.e. only the lower wall moves while the upper wall is stationary. The value for $\mathscr{R}$ is, of course, unchanged if the upper wall oscillates while the lower wall is stationary. The general case when Λ_x and Λ_y are non-zero and differ in magnitude and when the phase angles σ, γ_0 and γ_1 all differ is complicated, and it does not seem possible to reduce the results to a simple form. Two special cases, however, can be so reduced and indicate what might be expected for the general case. In both cases, we set $\Lambda_x = \Lambda_y = \Lambda$. For case (i), the top wall oscillates in each direction in phase with the lower wall, and so, with reference to (2.3a,b) and (2.4a,b), we take $\sigma = 0$, $\gamma_1 = \gamma_0$. For case (ii), the top wall is exactly 180° out of phase with the bottom wall, and so $\sigma = \pi$, $\gamma_1 = \gamma_0 + \pi$. For either case, Ra_2 can still be expressed in the form (3.12), so that all the above comments concerning the effect of γ_0 and λ for the case when the upper wall is at rest are still applicable. The definition of $\mathscr{R}$, however, is now different but is rather lengthy and will not be given explicitly. Suffice it to say that, for case (i), $\mathscr{R} \to 0$ as $\beta \to 0$ because we have oscillating plug flow with zero shear in this limit. This means that for case (i) there are additional terms in the definition of $\mathscr{R}$ which must be subtracted from those given on the right in (3.11), at least in the limit $\beta \to 0$. For case (ii), these same terms change sign and so add to the terms in (3.11). Thus, an out-of-phase oscillation of 180° between the upper and lower walls is more stabilizing than having only the lower wall oscillating as one might expect.

Numerical results for Ra_2 will be given in §5, after the limit $\beta \to 0$ is discussed.

4. The low-frequency limit

As $\beta \to 0$, the expansion (3.1a–c) is no longer suitable from a numerical viewpoint as a means for calculating Ra_2 because certain individual terms in (3.11) diverge, although the factor $\mathscr{R}$ certainly has a limiting value related, as we shall show later, to the degree of stabilization of transverse rolls due to a steady unidirectional shear. The low-frequency expansion done in this section not only eliminates this difficulty but also gives additional insight into the nature of the disturbed flow in this limit. A brief summary of this analysis and the corresponding results has been given by Kelly (1992). First we define

$$W(z,t) = W^*(z,t)\exp\left(\int^t \Phi(\hat{\tau})\,\mathrm{d}\hat{\tau}/2\beta^2\right) \stackrel{\text{def}}{=} W^*E(t), \quad \Theta(z,t) = \Theta^*(z,t)E(t). \quad (4.1\,a,b)$$

Upon substitution into (2.21a,b), we obtain

$$\left\{\Phi(t) + \mathrm{i}k_x Re_x U + \mathrm{i}k_y Re_y V - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\right\}\left(\frac{\partial^2}{\partial z^2} - k^2\right)W^*$$
$$+ 2\beta^2\left(\frac{\partial^2}{\partial z^2} - k^2\right)\frac{\partial W^*}{\partial t} - \mathrm{i}k_x Re_x \frac{\partial^2 U}{\partial z^2}W^* - \mathrm{i}k_y Re_y \frac{\partial^2 V}{\partial z^2}W^* = -k^2 Ra\,\Theta^*, \quad (4.2\,a)$$

$$\left\{Pr\,\Phi(t) + \mathrm{i}k_x Re_x Pr\,U + \mathrm{i}k_y Re_y Pr\,V - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\right\}\Theta^* + 2\,Pr\,\beta^2\frac{\partial \Theta^*}{\partial t} = W^*. \quad (4.2\,b)$$

If we again let $Re_x = Re$, $\lambda = Re_y/Re_x$, we conclude from (4.2a,b) that $\Phi(t) \sim O(Re)$ if $\lambda \sim O(1)$ and if we expand about the neutral state where $Ra = Ra_{c,0}$. From the definition of E in (4.1a), we expect $E \sim O(1)$ when $Ra = Ra_c$ for $0 \leqslant Re \ll 1$, and so we conclude that we should consider the case when $Re \sim O(\beta^2)$. For simplicity, we define now $\beta^2 = \delta$ and expand in terms of δ as

$$\begin{Bmatrix} W^* \\ \Theta^* \\ \Phi \\ \phi_0 \\ Ra_c \end{Bmatrix} = \begin{Bmatrix} W_0^* \\ \Theta_0^* \\ \Phi_0 \\ \phi_{00} \\ Ra_{c,0} \end{Bmatrix} + \delta \begin{Bmatrix} W_1^* \\ \Theta_1^* \\ \Phi_1 \\ \phi_{01} \\ Ra_{c,1} \end{Bmatrix} + \delta^2 \begin{Bmatrix} W_2^* \\ \Theta_2^* \\ \Phi_2 \\ \phi_{02} \\ Ra_{c,2} \end{Bmatrix} + \dots. \qquad (4.3a\text{–}e)$$

We also define $Re = \delta Re_1$. Because we are expanding about a neutral state for which the principle of exchange of stabilities holds, we set $\Phi_0 \equiv 0$. The equations for W_0^* and Θ_0^* are the same as those for W_0 and Θ_0, namely, (3.2a,b). We set $k = k_c$ and consider (again for simplicity in the presentation of the analysis) only case I when $A_x = A_y = 0$. The terms corresponding to the low-frequency analysis of ϕ_0 are as follows:

$$\phi_{00}(z) = 1-z, \quad \phi_{01}(z) = \tfrac{1}{6}\mathrm{i}(1-z)(z^2-2z) \overset{\text{def}}{=} \tfrac{1}{6}\mathrm{i}f_1 \qquad (4.4a,b)$$

(when $Re \sim O(\beta^2)$, ϕ_{02} is not required to determine $Ra_{c,2}$). After substituting (4.3a–e) into (4.2a,b), we obtain at $O(\delta)$ the following equations:

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)^2 W_1^* - k^2 Ra_{c,0}\,\Theta_1^* = [\Phi_1 + \mathrm{i}\,Re_1\{k_x(1-z)\cos t + \lambda k_y(1-z)\cos(t+\gamma_0)\}]$$
$$\times\left(\frac{\partial^2 W_0^*}{\partial z^2} - k^2 W_0^*\right) + 2\left(\frac{\partial^2}{\partial z^2}-k^2\right)\frac{\partial W_0^*}{\partial t} + k^2 Ra_{c,1}\,\Theta_0^*, \qquad (4.5a)$$

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)\Theta_1^* + W_1^* = [Pr\,\Phi_1 + \mathrm{i}\,Re_1 Pr\{k_x(1-z)\cos t + \lambda k_y(1-z)\cos(t+\gamma_0)\}]\,\Theta_0^*$$
$$+2Pr\frac{\partial \Theta_0^*}{\partial t}. \qquad (4.5b)$$

We express $W_0^*(z,t)$ and $\Theta_0^*(z,t)$ as

$$W_0^*(z,t) = A_0(t)\,F_0(z), \quad \Theta_0^*(z,t) = A_0(t)\,G_0(z), \qquad (4.6)$$

where $F_0(z)$ and $G_0(z)$ satisfy the same ordinary differential equations as W_0^* and Θ_0^*. If we substitute (4.6) into (4.5a,b) and again develop a solvability condition as in §3, we obtain the following equation governing A_0 and Φ_1:

$$2\frac{\mathrm{d}A_0}{\mathrm{d}t}\left[\int_0^1 F_0(F_0''-k^2F_0)\,\mathrm{d}z - k^2 Ra_{c,0} Pr\int_0^1 G_0^2\,\mathrm{d}z\right]$$
$$+\Phi_1 A_0\left[\int_0^1 F_0(F_0''-k^2F_0)\,\mathrm{d}z - k^2 Ra_{c,0} Pr\int_0^1 G_0^2\,\mathrm{d}z\right]$$
$$+\mathrm{i}\,Re_1 A_0\left[\{k_x\cos t + \lambda k_y\cos(t+\gamma_0)\}\left\{\int_0^1 (1-z)\,F_0(F_0''-k^2F_0)\,\mathrm{d}z\right.\right.$$
$$\left.\left.-k^2 Ra_{c,0} Pr\int_0^1 (1-z)\,G_0^2\,\mathrm{d}z\right\}\right] + k^2 Ra_{c,1} A_0\int_0^1 F_0 G_0\,\mathrm{d}z = 0. \qquad (4.7)$$

If we consider for the moment that case when $Re_1 = 0$ and $Ra > Ra_{c,0}$, then either the term involving $\mathrm{d}A_0/\mathrm{d}t$ or that involving Φ_1 can be used to calculate the linear growth rate due to our scaling. In (4.7), the term involving Re_1 is imaginary, and we balance it by the imaginary part of Φ_1, namely, $\Phi_{1\mathrm{i}}$. We can then satisfy (4.7) by taking $A_0 = \text{constant} = a_0$, $\Phi_{1\mathrm{r}} = 0$, $Ra_{c,1} = 0$, and

$$\Phi_{1\mathrm{i}} = \frac{-C_2 Re_1\{k_x \cos t + \lambda k_y \cos(t+\gamma_0)\}}{C_1}, \tag{4.8}$$

where
$$C_1 = \int_0^1 F_0(F_0'' - k^2 F_0)\,\mathrm{d}z - k^2 Pr\, Ra_{c,0} \int_0^1 G_0^2\,\mathrm{d}z \tag{4.9a}$$

and
$$C_2 = \int_0^1 (1-z) F_0(F_0'' - k^2 F_0)\,\mathrm{d}z - k^2 Pr\, Ra_{c,0} \int_0^1 (1-z) G_0^2\,\mathrm{d}z. \tag{4.9b}$$

If the analysis were to be repeated for the case of a steady shear flow in, say, the x-direction for disturbances with $k_y = 0$, then we would have $\beta^2 = Re$ (so $Re_1 = 1$), and we would find that $\Phi_{1\mathrm{i}} = -C_2 k_x / C_1$, so that (C_2/C_1) corresponds to a wave velocity in the x-direction. Thus, $\Phi_{1\mathrm{i}}$ represents a phase function for the unsteady problem. Note that $t = \omega^* t^* = 2\beta^2 Pr\,\tau$, where τ is a time variable based on the diffusion timescale h^2/κ_0; hence, $t \sim O(\beta^2)$ for $\beta^2 \ll 1$ if the diffusion timescale is used.

With A_0 and Φ_1 determined so as to satisfy the solvability condition, then we can say that a solution exists at $O(\delta)$ of the form

$$W_1^*(z,t) = \mathrm{i}Re_1\{k_x \cos t + \lambda k_y \cos(t+\gamma_0)\}\, a_0 F_1(z), \tag{4.10a}$$
$$\Theta_1^*(z,t) = \mathrm{i}Re_1\{k_x \cos t + \lambda k_y \cos(t+\gamma_0)\}\, a_0 G_1(z), \tag{4.10b}$$

where the equations for F_1 and G_1 are omitted for brevity.

We now go on to $O(\delta^2)$ in order to determine $Ra_{c,2}$. The equations for W_2^* and Θ_2^* are, after substituting for W_1^* and Θ_1^*,

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 W_2^* - k^2 Ra_{c,0}\,\Theta_2^* &= \Phi_2 (F_0'' - k^2 F_0)\, a_0 \\ &\quad - Re_1^2\{k_x \cos t + \lambda k_y \cos(t+\gamma_0)\}^2 \left(1 - z - \frac{C_2}{C_1}\right)(F_1'' - k^2 F_1)\, a_0 \\ &\quad - 2\mathrm{i}\, Re_1\{k_x \sin t + \lambda k_y \sin(t+\gamma_0)\}(F_1'' - k^2 F_1)\, a_0 \\ &\quad - \tfrac{1}{3}\mathrm{i}\, Re_1\{k_x \sin t + \lambda k_y \sin(t+\gamma_0)\}\{f_1(F_0'' - k^2 F_0) - f_1'' F_0\}\, a_0 + k^2 Ra_{c,2} G_0 a_0,\end{aligned} \tag{4.11a}$$

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} - k^2\right)\Theta_2^* + W_2^* &= Pr\,[\Phi_2 G_0 a_0 - Re_1^2\{k_x \cos t + \lambda k_y \cos(t+\gamma_0)\}^2 \left(1 - z - \frac{C_2}{C_1}\right) \\ &\quad \times G_1 a_0 - 2\mathrm{i}\, Re_1\{k_x \sin t + \lambda k_y \sin(t+\gamma_0)\} G_1 a_0 - \tfrac{1}{3}\mathrm{i}\, Re_1\{k_x \sin t + \lambda k_y \sin(t+\gamma_0)\} f_1 G_0 a_0].\end{aligned} \tag{4.11b}$$

If we multiply (4.11*a*) by F_0, (4.11*b*) by $(-k^2 Ra_{c,0} G_0)$, integrate and add the results, the solvability condition is obtained as

$$\Phi_2\left\{\int_0^1 F_0(F_0''-k^2F_0)\,\mathrm{d}z-k^2 Pr\, Ra_{c,0}\int_0^1 G_0^2\,\mathrm{d}z\right\}+k^2 Ra_{c,2}\int_0^1 F_0G_0\,\mathrm{d}z-Re_1^2\{k_x\cos t$$
$$+\lambda k_y\cos(t+\gamma_0)\}^2\left\{\int_0^1\left(1-z-\frac{C_2}{C_1}\right)F_0(F_1''-k^2F_1)\,\mathrm{d}z-k^2 Pr\, Ra_{c,0}\int_0^1\left(1-z-\frac{C_2}{C_1}\right)G_0G_1\,\mathrm{d}z\right\}$$
$$-2\mathrm{i}Re_1\{k_x\sin t+\lambda k_y\sin(t+\gamma_0)\}\left\{\int_0^1 F_0(F_0''-k^2F_0)\,\mathrm{d}z-k^2 Pr\, Ra_{c,0}\int_0^1 G_0G_1\,\mathrm{d}z\right\}$$
$$-\tfrac{1}{3}\mathrm{i}Re_1\{k_x\sin t+\lambda k_y\sin(t+\gamma_0)\}\left\{\int_0^1 f_1F_0(F_0''-k^2F_0)\,\mathrm{d}z-\int_0^1 f_1''F_0^2\,\mathrm{d}z\right.$$
$$\left.-k^2 Pr\, Ra_{c,0}\int_0^1 f_1G_0^2\,\mathrm{d}z\right\}=0. \quad (4.12)$$

Now F_0, G_0, F_1, G_1, f_1 and C_2/C_1 are all real, and so it is clear that Φ_2 is complex. The imaginary part of Φ_2 gives a correction to the phase function arising from higher-order terms in the velocity profile as well as the time-derivatives of W_1^* and Θ_1^*. The real part of Φ_2 can be identified as variation with time of the amplitude of the disturbance and is determined by the relation

$$\Phi_{2r}\left\{\int_0^1 F_0(F_0''-k^2F_0)\,\mathrm{d}z-k^2 Ra_{c,0}\int_0^1 G_0^2\,\mathrm{d}z\right\}$$
$$+k^2 Ra_{c,2}\int_0^1 F_0G_0\,\mathrm{d}z-Re_1^2\{k_x\cos t+\lambda k_y\cos(t+\gamma_0)\}^2$$
$$\times\left\{\int_0^1\left(1-z-\frac{C_2}{C_1}\right)F_0(F_1''-k^2F_1)\,dz-k^2 Pr\, Ra_{c,0}\int_0^1\left(1-z-\frac{C_2}{C_1}\right)G_0G_1\,\mathrm{d}z\right\}=0. \quad (4.13)$$

Now some of the explicitly time-dependent terms in (4.13) are periodic and can be accommodated by choosing Φ_{2r} properly, which would also be periodic in time. However, a non-zero mean value of Φ_{2r} (>0) would give rise to unbounded growth in time, and so we must have $\bar{\Phi}_{2r}=0$, where the average is over one cycle of oscillation. When we use this average in (4.13), we obtain

$$k^2 Ra_{c,2}\int_0^1 F_0G_0\,\mathrm{d}z=\tfrac{1}{2}(k_x^2+2\lambda k_xk_y\cos\gamma_0+\lambda^2k_y^2)$$
$$\times\left\{\int_0^1\left(1-z-\frac{C_2}{C_1}\right)F_0(F_1''-k^2F_1)\,\mathrm{d}z-k^2 Pr\, Ra_{c,0}\int_0^1\left(1-z-\frac{C_2}{C_1}\right)G_0G_1\,\mathrm{d}z\right\}. \quad (4.14)$$

The conditions for which the term involving k_x and k_y in (4.14) is zero are discussed after (3.12); for $\lambda\neq0$ and $\gamma_0\neq0,\pi$ it is non-zero. For $\lambda=0$, the term is zero for longitudinal rolls ($k_x=0$). The conditions for which this term is an extremum are discussed after (3.13). P. Hall has pointed out to the authors that this result for $Ra_{c,2}$ holds also for the case when $Re\sim O(\beta)$.

The actual value of $Ra_{c,2}$ can be found by using established results for steady Couette flow (Ingersoll 1966) for $Re\ll1$ because we are considering the quasi-steady limit. Thus, if we consider the case of a steady shear in, say, the x-direction and consider only transverse disturbances ($k_x\neq0$, $k_y=0$), we let $\delta=Re$ in (4.3*a–e*) and expand in terms of Re for $Re\ll1$ when $U''=0$, where the Φ_j are now constants and $Ra_{c,2}\to(Ra_{c,2})_{\beta=0}$. The other variables are expanded in a similar manner. At lowest order, $\Phi_0=0$ whereas at $O(Re)$ we find that

$$\Phi_1=-\mathrm{i}k_x(C_2/C_1),$$

where C_1 and C_2 are the same as defined in (4.9*a*, *b*). For the case of steady shear, C_2/C_1 represents a wave velocity, as noted after (4.9). Continuing on to $O(Re^2)$, a solvability condition yields the result

$$(Ra_{c,2})_{\beta=0} = \frac{\int_0^1 \left(1-z-\frac{C_2}{C_1}\right) F_0(F_1''-k^2F_1)\,dz - k^2 Pr\, Ra_{c,0} \int_0^1 \left(1-z-\frac{C_2}{C_1}\right) G_0 G_1\,dz}{\int_0^1 F_0 G_0\,dz} \quad (4.15)$$

Now (4.14) is the same as (4.15) except for the factor of $(\frac{1}{2}K)$ which is equal to one-half if we take $k_y = 0$ so $k = k_x$. The factor of one-half occurs in (4.14) because the effectively steady shear there is equal to the mean-squared value of the oscillatory shear. Thus, we can say

$$(Ra_{c,2})_{0<\beta^2\ll 1} = \tfrac{1}{2}K(Ra_{c,2})_{\beta=0}, \quad (4.16)$$

where an explicit result for $(Ra_{c,2})_{\beta=0}$ on the right has been given by Ingersoll (1966) as follows for $k = k_{c,0}$:

$$(Ra_{c,2})_{\beta=0} = 0.5598Pr^2+0.1270Pr+0.06451. \quad (4.17)$$

Clearly, $(Ra_{c,2})_{\beta=0}$ is equal to the factor $\mathscr{R}$ in (3.12) as $\beta \to 0$. It is worth noting that Ingersoll's result predicts that $Ra_{c,2} > 0$ and that $Ra_{c,2}$ increases as Pr increases, which results are, of course, consistent with the results of §3. The result (4.16) allows us to predict $Ra_{c,2}$ for $\beta^2 \ll 1$ for general k_x, k_y, γ_0 and λ without further numerical integration. Together with the results of §3, we conclude that stabilization occurs in the quasi-steady limit $\beta^2 \to 0$ although it should be emphasized that we have taken $Re \sim O(\beta^2)$. The case when $\beta^2 \to 0$ with Re fixed so $0 < Re \ll 1$ requires a separate analysis.

For the case when both walls oscillate so that the shear of the basic flow is non-planar as $\beta \to 0$, it is not surprising that Ra_c is changed for $Re > 0$. The above analysis indicates, however, that the same result holds when only one wall oscillates (i.e. $A_x = A_y = 0$), which might seem to be surprising because it would seem that the shear would then instantaneously be in one direction. If so, one might expect rolls to form whose axes would change in time so as to point in this direction. However, the boundary condition at the stationary upper wall would cause a shear to be exerted so as to oppose this change in orientation. Actually, the basic shear even for this case is non-planar for any $\beta > 0$ which can be seen by considering the special case $\lambda = 1$ and $\gamma_0 = \pm\frac{1}{2}\pi$. For this case, the motion of the lower wall is equivalent to a rotation of the velocity with angular velocity ω about the z-axis, and so one sees the lower wall moving in a fixed direction if one adopts a coordinate system rotating with this angular velocity. However, the upper wall is then seen as rotating in the opposite direction, thereby creating non-planar shear.

5. Numerical results

Numerical values of $\mathscr{R}$ as defined by (3.11) and (3.12) as a function of β and Pr will now be presented. These were obtained by first evaluating the functions F_{11}, etc. by a shooting method employing a fourth-order Runge–Kutta scheme and then using Simpson's one-third rule for quadrature.

The results for oscillations of the lower wall only (i.e. $A_x = A_y = 0$) are presented in figure 1. It was found that the data could be shown on a single diagram by plotting

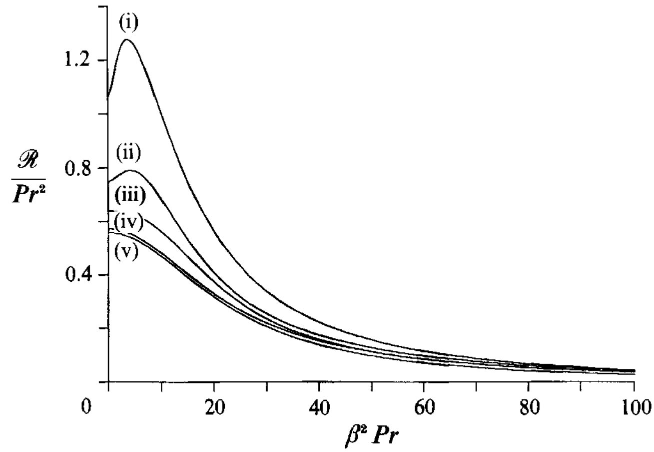

FIGURE 1. The stabilization factor $\mathscr{R}$ for non-planar oscillations of one wall as a function of the non-dimensional frequency β. (i) $Pr = 0.5$, (ii) $Pr = 1$, (iii) $Pr = 2$, (iv) $Pr = 10$, (v) $Pr = 1000$.

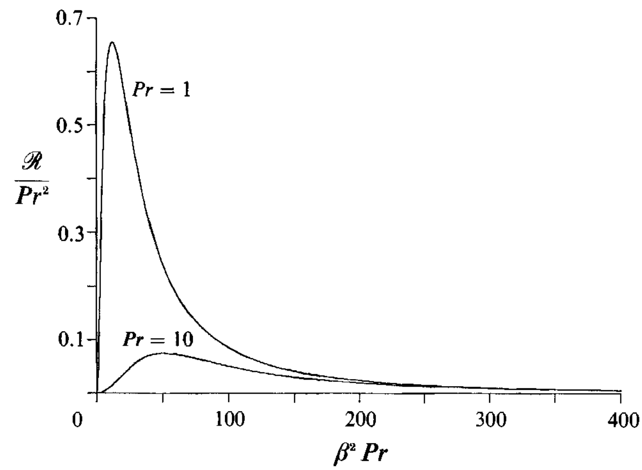

FIGURE 2. The stabilization factor $\mathscr{R}$ for transverse disturbances due to two walls oscillating in phase as a function of the non-dimensional frequency β.

$\mathscr{R}/Pr^2$ as a function of $\beta^2 Pr$. The former quantity was suggested by the quasi-steady result (4.16), (4.17) for $Pr \gg 1$, whereas the latter quantity is the effective non-dimensional frequency in the energy equation (2.23*b*). Although the maximum value of $\mathscr{R}/Pr^2$ decreases as Pr increases, the actual value of $\mathscr{R}$ increases with Pr in accordance with the results of the previous section. The range of $\beta^2 Pr$ over which $\mathscr{R}/Pr^2$ is significantly greater than zero is almost independent of Pr, indicating that the effect is correlated with the ratio of the unsteady thermal boundary-layer thickness associated with the interaction to the depth. For $Pr > 2$, maximum stabilization occurs as $\beta \to 0$ and the amount of stabilization is in agreement with the quasi-steady result. For values of Pr less than two, however, the curves are not monotonic and the maximum degree of stabilization occurs at some $\beta > 0$.

The results for stabilization of transverse rolls when both walls oscillate in phase in the x-direction are shown in figure 2 for $Pr = 1$ and 10. For this case, of course, stabilization is not realizable because longitudinal rolls become unstable when $Ra > Ra_{c,0}$. In the light of the comments made at the end of §3, however, the results are characteristic of the non-planar case when stabilization is achieved if the upper and

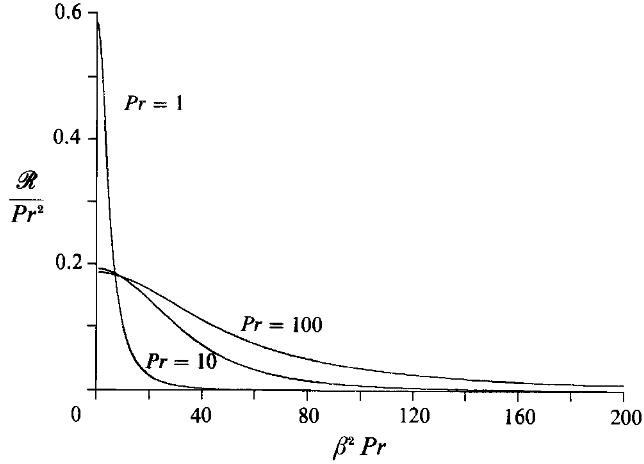

FIGURE 3. The stabilization factor $\mathscr{R}$ for non-planar oscillatory flow due to an oscillatory pressure gradient as a function of the non-dimensional frequency β.

lower walls have velocity components in the x- and y-directions that are in phase (although the x-component and y-component are not in phase or directly out of phase, i.e. $\gamma_0 \neq 0, \pi$). For this case, the amount of stabilization tends to zero as $\beta \rightarrow 0$ as well as when $\beta \rightarrow \infty$, because the shear vanishes as $\beta \rightarrow 0$. Thus, the convection cells are not tilted in this limit. Therefore, it is not surprising that a maximum degree of stabilization occurs at a non-zero value of β for this case. Although the amount of stabilization achieved in this case for $Pr = 1$ is comparable to that shown in figure 1, curve (ii), the amount achieved when $Pr = 10$ is an order of magnitude less than that shown in figure 1, curve (iv) for which the greatest stabilization occurs as $\beta \rightarrow 0$.

Finally, the results for pulsating Poiseuille flow are shown in figure 3. For both low and high values of Pr, the maximum degree of stabilization occurs as $\beta \rightarrow 0$. In this limit, the numerical results agree with those given in Appendix A of the paper by Müller, Lücke & Kamps (1992). However, the correlation of the results with the parameter $\beta^2 Pr$ is not as good for Poiseuille flow as for Couette flow (figures 1, 2). In constructing figure 3, the maximum velocity as $\beta \rightarrow 0$, namely, $\frac{1}{8}\bar{U}_2^*$ was used as a reference velocity in order to have the amount of stabilization occurring in the quasi-steady limit to be comparable to that shown in figure 1.

6. Summary and conclusions

Although oscillatory flow alone one axis serves as a wavenumber selection mechanism just as a steady shear flow, non-planar oscillations have been shown to have a stronger effect upon the onset of thermal convection. They not only act, at least on the basis of linear theory, as a pattern selection mechanism, but also actually stabilize the system, i.e. the critical Rayleigh number is greater (in general) when non-planar oscillations occur than when they do not. The effect seems to be associated strongly with the stabilizing effect of a steady non-planar shear flow upon arbitrarily orientated disturbances. In general, the amount of stabilization is greatest as the non-dimensional frequency parameter tends to zero and as the Prandtl number becomes large.

The actual amount of stabilization shown in figures 1–3 is rather slight for the small values of Reynolds numbers pertinent to this analysis. However, the same is true for the case of a steady shear. But it is known that a steady shear can profoundly affect the value of the critical Rayleigh number for transverse disturbances when values of Re of $O(1)$ or more are considered. For instance, for Couette–Poiseuille flow, Fujimura & Kelly (1988) found that Ra_c for transverse disturbances was between six and ten times the value without shear even for a *low* Prandtl number fluid ($Pr = 0.51$) when the characteristic Reynolds number was equal to 100 (which, after all, is not all that large as far as Reynolds number go; transition for steady planar Couette seems to occur at $Re \approx 1000$ (Reichart 1959) and for an unbounded unsteady Stokes layer at $Re \approx 600/\beta$ (Monkewitz & Bunster 1987)). It remains to be seen whether or not similar stabilization occurs for the case of non-planar oscillations when Re is large but, in view of the low-frequency results obtained above for small values of Re, there is every reason to be optimistic.

In considering the possibility of stabilization, we have to bear in mind the fact that only a linear analysis has been made here. We therefore have to assume that a supercritical bifurcation occurs, which is indeed the case for steady Couette and Poiseuille flow when heated from below (Clever, Busse & Kelly 1977 and Clever & Busse 1991). Hence, there is reason to expect a supercritical bifurcation for low values of β. The situation is basically different from the case of temperature modulation for which Roppo *et al.* (1984) have found a subcritical instability at low values of β. In their case, modulation of the temperature difference meant that symmetry of the basic state was lost, and so subcritical instability came perhaps as no great surprise. In the present case, however, the basic temperature *always* exhibits a linear variation with z, and so there is no obvious reason to expect a subcritical instability.

Various applications of the above concept to other problems concerning thermal convection can be made. For instance, the occurrence of thermal convection in a shear flow has importance for horizontal chemical vapour deposition reactors with through-flow, as discussed by Evans & Grief (1989). The Reynolds number associated with the mean flow is usually moderate (~ 20). For that case, the mean shear itself stabilizes transverse disturbances and so it is sufficient to impose only an oscillation in the cross-flow direction in order to stabilize the system as a whole. For small but comparable values of the Reynolds numbers associated with the mean shear Re_0 (assumed, say, to be in the y-direction) and the oscillation Reynolds number Re_x (associated, say, with an oscillation in the x-direction) the results given by (3.12) for $\lambda = 0$ and (4.17) can be combined to yield the relation

$$Ra_c - Ra_{c,0} = \tfrac{1}{2}Re_x^2 \cos^2\theta\, \mathscr{R}(Pr, \beta) + Re_0^2 \sin^2\theta\, Q(Pr), \qquad (6.1)$$

where $Q(Pr)$ is defined by (4.17). It should be worth noting that this result holds only if the imposed frequency differs from the intrinsic frequency associated with any mean convective effect, which case must be investigated separately. The factor in $\sin^2\theta$ in (6.1) comes from the term $(k_y/k)^2$ which arises from the generalization of (4.17) to a roll with arbitrary orientation. For $\beta \rightarrow 0$, it has already been argued that $\frac{1}{2}\mathscr{R} \rightarrow Q$ and so, in the quasi-steady limit, the minimum value of $(Ra_c - Ra_{c,0})$ is obtained in the same way as the minimum value of K was determined from (3.14), with naturally the same conclusion. For $Re_x < Re_0$, longitudinal rolls in the y-direction are preferred whereas for $Re_x > Re_0$ longitudinal rolls in the x-direction are preferred. In either case $Ra_c > Ra_{c,0}$.

The concept of using non-planar flow oscillations to stabilize the fluid state might be applicable to certain other fluid stability problems. For instance Takeuchi & Jankowski (1981) and Ng & Turner (1982) have shown that a steady axial flow tends to stabilize Taylor vortices occurring between concentric rotating cylinders. In view of the quasi-steady result obtained here, there is reason to expect stabilization also for the case, say, when one cylinder oscillates in the axial direction about a zero mean.

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Appendix. The disturbance mechanical energy equation for the case of planar oscillations

In order to gain insight into the mechanism of stabilization, the mechanical energy equation for a transverse roll acted upon by a planar oscillation in the x-direction has been investigated. For this case, of course, transverse rolls are not the most unstable type of disturbance. However, it should now be clear that the mechanism causing the factor $\mathscr{R}$ in (3.12) to be greater than zero is similar to the mechanism causing transverse rolls to be stabilized by a planar oscillation, and so the following results are certainly relevant to the non-planar case. For steady flow, the mechanism of stabilization has been discussed by Asai (1970) and Lipps (1971).

If we start with the linearized form of the momentum equations in the (x, z)-plane, multiply the x-momentum equation by u, the z-momentum equation by w, add the results and integrate over a wavelength and the fluid depth, we obtain the mechanical energy equation for the disturbance. In non-dimensional form, it is

$$2\beta^2 \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \langle \tfrac{1}{2}(u^2+w^2)\rangle \,\mathrm{d}z = Re\int_0^1 \left\langle -uw\frac{\partial U}{\partial z}\right\rangle \mathrm{d}z + Ra\int_0^1 \langle \theta w\rangle \,\mathrm{d}z - \int_0^1 \left\langle \left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)^2\right\rangle \mathrm{d}z, \tag{A 1}$$

where $\langle \ldots \rangle$ denotes an average over a wavelength. The first integral on the right-hand side of (A 1) represents the rate of transfer of energy between the basic flow and the disturbance, the second integral represents the rate of release of buoyant energy, and the third integral represents the rate at which energy is dissipated by viscosity and is always positive. We will be interested in determining how the integrands of these three integrals depend upon z.

Again assuming that $Re \ll 1$, we expand u in powers of Re along with w and θ, as done in §3, except that the terms representing non-planar effects (e.g. W_{12} in (3.4a)) are now taken to be zero. Once these expansions are substituted into the integrands, we can express the integrands as follows:

$$Re\left\langle -uw\frac{\partial U}{\partial z}\right\rangle = Re\{r_1(z)\,\mathrm{e}^{\mathrm{i}t}+\hat{r}_1(z)\,\mathrm{e}^{-\mathrm{i}t}\}+Re^2\{\bar{r}_2(z)+r_2(z)\,\mathrm{e}^{2\mathrm{i}t}+\hat{r}_2(z)\,\mathrm{e}^{-2\mathrm{i}t}\}+O(Re^3), \tag{A 2a}$$

$$Ra\,\langle \theta w\rangle = Ra\,[b_0(z)+Re\{b_1(z)\,\mathrm{e}^{\mathrm{i}t}+\hat{b}_1(z)\,\mathrm{e}^{-\mathrm{i}t}\} + Re^2\{\bar{b}_2(z)+b_2(z)\,\mathrm{e}^{2\mathrm{i}t}+\hat{b}_2(z)\,\mathrm{e}^{-2\mathrm{i}t}\}+O(Re^3)]. \tag{A 2b}$$

$$\left\langle \left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)^2\right\rangle = d_0(z)+Re\{d_1(z)\,\mathrm{e}^{\mathrm{i}t}+\hat{d}_1(z)\,\mathrm{e}^{-\mathrm{i}t}\} + Re^2\{\bar{d}_2(z)+d_2(z)\,\mathrm{e}^{2\mathrm{i}t}+\hat{d}_2(z)\,\mathrm{e}^{-2\mathrm{i}t}\}+O(Re^3). \tag{A 2c}$$

Because the change in the value of Ra_c due to the oscillation is of $O(Re^2)$, we are

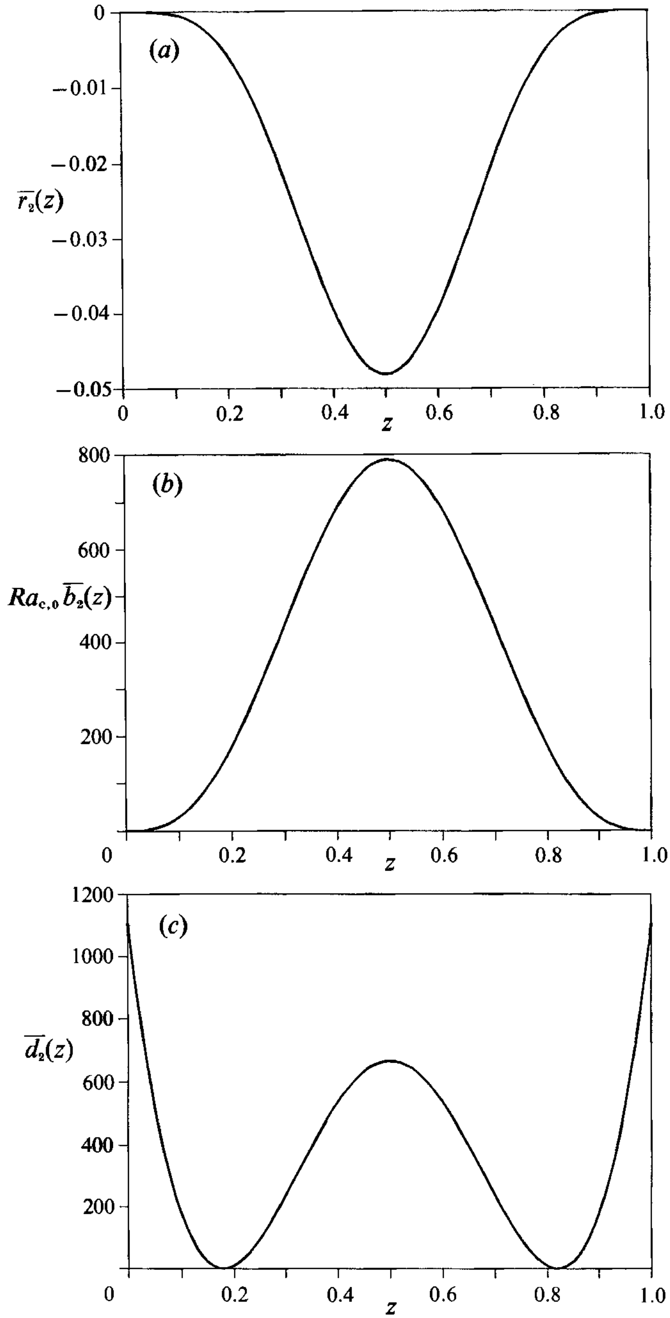

FIGURE 4. (*a*) The integrand $\bar{r}_2(z)$ of the term in the mechanical energy term representing the mean rate of energy transfer between the basic flow and the disturbance due to an oscillation of the lower wall. (*b*) The integrand $Ra_{c,0}\bar{b}_2(z)$ of the term in the mechanical energy representing the contribution to the mean rate of buoyant energy generation due to the oscillation of the lower wall. (*c*) The integrand $\bar{d}_2(z)$ of the contribution to the mean viscous dissipation due to the oscillation of the lower wall. $Pr = 10$, $\beta = 0.5$.

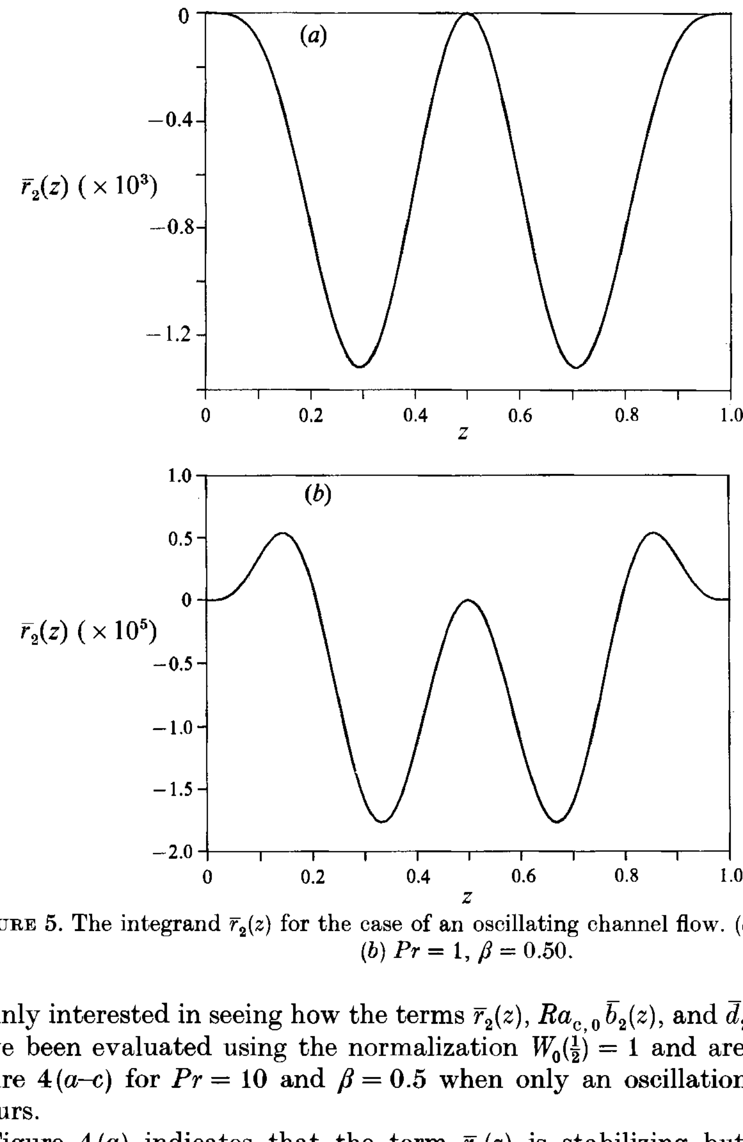

FIGURE 5. The integrand $\bar{r}_2(z)$ for the case of an oscillating channel flow. (*a*) $Pr = 100$, $\beta = 0.20$; (*b*) $Pr = 1$, $\beta = 0.50$.

mainly interested in seeing how the terms $\bar{r}_2(z)$, $Ra_{c,0}\,\bar{b}_2(z)$, and $\bar{d}_2(z)$ compare. These have been evaluated using the normalization $W_0(\frac{1}{2}) = 1$ and are shown in order in figure 4(*a–c*) for $Pr = 10$ and $\beta = 0.5$ when only an oscillation of the lower wall occurs.

Figure 4(*a*) indicates that the term $\bar{r}_2(z)$ is stabilizing but its importance is negligible compared to $Ra_{c,0}\,\bar{b}_2(z)$ and $\bar{d}_2(z)$ as indicated by figures 4(*b*) and 4(*c*). Figure 4(*b*) indicates that the change in the buoyant release of energy $\bar{b}_2(z)$ actually is positive (i.e. it is destabilizing) due to the oscillation. However, as figure 4(*c*) indicates, the dissipation function increases even more and appears to be the major stabilizing factor for this case. Similar trends occur for $Pr = 1000$ when $\beta = 0.05$. It should be mentioned that the integrated values of these functions have been checked to give the result that the increase in mean disturbance mechanical energy is zero when $Ra = Ra_c$.

For the case of pulsating Poiseuille flow, the integrands of the buoyancy and dissipation terms are qualitatively similar to those shown for Couette flow in figure 4. However, the integrand of the term involving the basic shear differs, as shown in figure 5(*a*, *b*). Numerical maxima now occur away from the mid-channel point. Furthermore, as figure 5(*b*) indicates, both positive and negative maxima can occur for the low-*Pr* case. Although the integrand is not entirely of one sign, the integrated effect still tends to be stabilizing. It will be of interest to see if this result occurs also for finite-amplitude flow oscillations.

REFERENCES

ASAI, T. 1970 Three dimensional features of thermal convection in a plane Couette flow. *J. Met. Soc. Japan* **48**, 18–29.

BARENGHI, C. F. & JONES, C. A. 1989 Modulated Taylor–Couette flow. *J. Fluid Mech.* **208**, 127–160.

CLEVER, R. M. & BUSSE, F. H. 1991 Instabilities of longitudinal rolls in the presence of Poiseuille flow. *J. Fluid Mech.* **229**, 517–529.

CLEVER, R. M., BUSSE, F. H. & KELLY, R. E. 1977 Instabilities of longitudinal convection rolls in Couette flow. *Z. Angew. Math. Phys.* **29**, 771–783.

DAVIS, S. H. 1976 The stability of time-periodic flows. *Ann. Rev. Fluid Mech.* **8**, 57–74.

DONNELLY, R. J. 1990 Externally modulated hydrodynamic system. In *Nonlinear Evolution of Spatio-Temporal Structures in Dissipative Continuous Systems* (ed. F. H. Busse & L. Kramer), pp. 31–43. Plenum.

EVANS, G. & GRIEF, R. 1989 A study of traveling wave instabilities in a horizontal channel flow with applications to chemical vapor deposition. *Intl J. Heat Mass Transfer* **32**, 895–911.

FINUCANE, R. G. & KELLY, R. E. 1976 Onset of instability in a fluid layer heat sinusoidally from below. *Int. J. Heat Mass Transfer* **19**, 71–85.

FUJIMURA, K. & KELLY, R. E. 1988 Stability of unstably stratified shear flow between parallel plates. *Fluid Dyn. Res.* **2**, 281–292.

INGERSOLL, A. P. 1966 Convective instabilities in plane Couette flow. *Phys. Fluids* **9**, 682–689.

KELLY, R. E. 1992 Stabilization of Rayleigh–Bénard convection by means of a slow nonplanar oscillatory flow. *Phys. Fluids* A **4**, 647–648.

KELLY, R. E. & THOMPSON, L. N. 1988 The onset of Rayleigh–Bénard convection in oscillatory Couette flow. In *AIAA/ASME/SIAM/APS 1st. National Fluid Dyn. Cong.: Collection of Technical Papers*, Part 3, 1765–1767.

LIPPS, F. B. 1971 Two-dimensional numerical experiments in thermal convection with vertical shear. *J. Atmos. Sci.* **28**, 3–19.

MEYER, C. W., CANNELL, D. S. & AHLERS, G. 1992 Hexagonal and roll flow patterns in temporally modulated Rayleigh–Bénard convection. *Phys. Rev.* A **45**, 8583–8604.

MONKEWITZ, P. A. & BUNSTER, A. 1987 The stability of the Stokes layer: visual observations and some theoretical considerations. In *Stability of Time Dependent and Spatially Varying Flows* (ed. D. L. Dwoyer & M. Y. Hussain). Springer.

MÜLLER, H. W., LÜCKE, M. & KAMPS, M. 1992 Transversal convection patterns in horizontal shear flow. *Phys. Rev.* A **45**, 3714–3726.

NG, B. S. & TURNER, E. R. 1982 On the linear stability of spiral flow between rotating cylinders. *Proc. R. Soc. Lond.* A **382**, 83–102.

REICHART, H. 1959 *Gesetzmässigkeiten der Geradlingen Turbulenten Couttesströmung.* Mitt. Max-Planck-Institute für Stromungsforschung, Göttingen, No. 22.

ROPPO, M. N., DAVIS, S. H. & ROSENBLAT, S. 1984 Bénard convection with time periodic heating. *Phys. Fluids* **27**, 796–803.

TAKEUCHI, D. I. & JANKOWSKI, D. F. 1981 A numerical and experimental investigation of the stability of spiral Poiseuille flow. *J. Fluid Mech.* **102**, 101–126.